Energy flow lines and the spot of Poisson-Arago

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Abstract

We show how energy flow lines answer the question about diffraction phenomena presented in 1818 by the French Academy: "*deduce by mathematical induction, the movements of the rays during their crossing near the bodies*". This provides a complementary answer to Fresnel's wave theory of light. A numerical simulation of these energy flow lines proves that they can reach the bright spot of Poisson-Arago in the shadow center of a circular opaque disc. For a monochromatic wave in vacuum, these energy flow lines correspond to the diffracted rays of Newton's *Opticks*.

1 Introduction

The answer Fresnel provided in 1818 in response to the French Academy's competition marks the beginning of the refutation of Newton's corpuscular theory of light and the rehabilitation of the Huygens wave theory. The competition topic was presented as follows:

"*...diffraction phenomena have been a subject of research for many physicists...but research has not yet sufficiently determined the movement of the rays near the body where the change occurs...it important...to further study...the physical manner in which rays are inflected and separated into different bands ... As a result the Academy is proposing this research...to be presented as follows: 1. Determine all the effects of ray diffraction ...direct and reflected when they ... pass near the extremities of a body... 2. Deduce from these experiments, by mathematical induction, the movements of the rays during their crossing near the bodies.*" [1]

This announcement was made by a jury of great scientists: Pierre-Simon Laplace, Jean B. Biot, Simeon D. Poisson, Joseph L. Gay-Lussac - all Newtonians - as well as Dominique F. Arago, who was the only one who believed in wave theory.

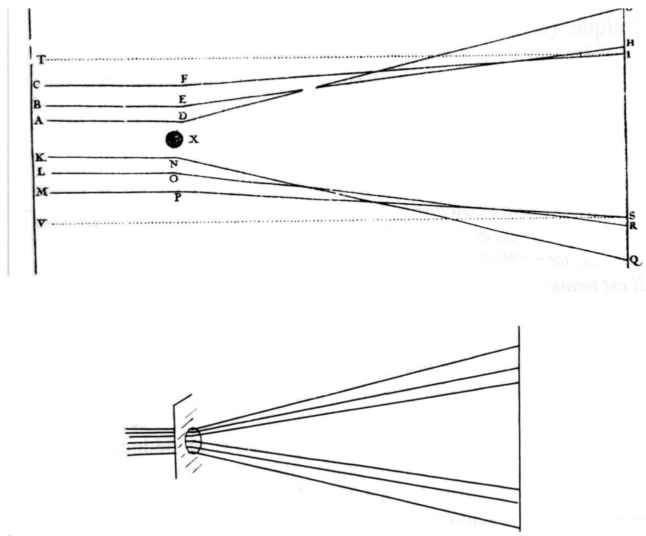

Figure 1: Newton's rays diffracted by a hair and a small circular aperture (1704). [2]

They all recall the ray concept proposed by Newton in the third book of his *Opticks* [2] [see Fig. 1] in order to explain diffraction by a hair or by a circular aperture, and the conclusion of its experimental part where he wrote:

"*When I made the foregoing Observations, I design'd to repeat most of them with more care and exactness, and to make some new ones for determining the manner how the Rays of Light are bent in their passage by Bodies, for making the Fringes of Colours with the dark lines between them. But I was then interrupted, and cannot now think of taking these things into farther Consideration. And since I have not finish'd this part of my Design, I shall conclude with proposing only some Queries, in order to a further search to be made by others.*" [2]

Fresnel's essay develops a mathematical wave theory which seems be in conflict with the corpuscular theory. It describes an impressive number of diffraction experiments all explained by the same principle: the fringes are due to interference waves issued by each of the screen points. Fresnel's principle generalizes Huygens' principle.

Poissson carefully studied Fresnel's theory and deduced "*that the center of the shadow of a circular opaque disc... (should)... be as enlightened as if the disc didn't exist*" [1]; this bright spot of light at the center of the shadow, he claimed, "*violated common sense*" and hence refuted Fresnel's wave theory. However, Arago almost immediatly verified the existence of the spot experimentally. Fresnel won the competition and this discovery marks the beginning of the acceptance of wave theory and the refutation of corpuscular theory. This spot of light, today known as *Poisson's bright spot* or *spot of Arago*, had been observed a century earlier (1723) by Maraldi, who had not published his work. [3, 4]

This paper proposes to complete Fresnel's answer and to show how the energy flow lines are (in the special case of a monochomatic wave in vacuum) the answer to the French Academy's question about diffraction phenomena:" *deduce by mathematical induction, the movements of the rays during their crossing near the bodies"*. We study, by a numerical simulation, the case of diffraction by a circular aperture as well as diffraction by a circular opaque disc in order to find the spot of Poisson-Arago. In Section 2, we recall how to calculate bright densities with wave theory. In Section 3, we show that the energy flow lines can correspond to Newtonian rays. Then in section 4, we discuss the interpretation of these energy flow lines.

2 Intensity distributions

Let us consider a monochromatic plane wave of light whose wavefronts lie in the plane of a circular aperture (resp. an opaque disc) placed in the xy plane, and a detector in a parallel plane at distance z. Set (x_M, y_M) the coordinates of a point M in the diffracting plane and (x, y) the coordinates of the observation point P on the detector.

First, if we neglect the polarisation of light and if we suppose that the incident wave is of the form $A_0 e^{ikz}$ on the aperture, the amplitude $A(P)$ for $z > 0$, which verifies the Helmholtz equation, is given by the Rayleigh-Sommerfeld formula: [5, 6]

$$A(P) = -\frac{iA_0}{\lambda}\int_S \frac{e^{ikr}}{r}(1 - \frac{1}{ikr})\cos\theta dx_M dy_M \qquad (1)$$

where $r = \sqrt{(x - x_M)^2 + (y - y_M)^2 + z^2}$, $\cos\theta = \frac{z}{r}$, $k = \frac{2\pi}{\lambda}$ and where the integration is taken on the surface S of the aperture. Notice that the formula gives a more exact solution, in particular for a very small distance of the aperture thanks to $-1/ikr$; see Fig. 2 of Gillen and Guha. [7]

2.1 Intensity distributions for a circular aperture

Figure 2 shows, in a plane (z,x) containing the optical axis, intensity behind a circular aperture with a radius $R = 5\mu m$ of a monochromatic plane wave of light with wavelength $\lambda = \frac{R}{10} = 0.5\mu m$.

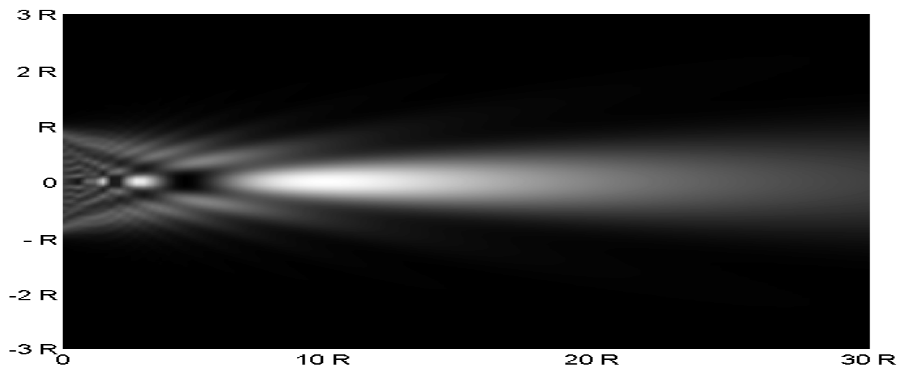

Figure 2: Calculated intensity distributions behind the circular aperture in the plane (z, x).

Far from the aperture, the classical Fraunhofer diffraction appears, and the emerging beam has a well-defined angular dispersion, in the order of $\Delta\theta \sim \frac{\lambda}{R}$; this is the Airy disc. Near the aperture, we get the Fresnel diffraction. We note a succession of bright and dark areas on the axis.

2.2 Intensity distributions for an circular opaque disk

Because the incident wave is a plane wave, the intensity value is calculated by the Babinet's principle [8, 9] in taking the square of

$$A(P) = A_0(e^{ikz} + \frac{i}{\lambda}\int_S \frac{e^{ikr}}{r}(1 - \frac{1}{ikr})\cos\theta dx_M dy_M); \quad (2)$$

numerical integration is taken on the surface of the opaque disc S.

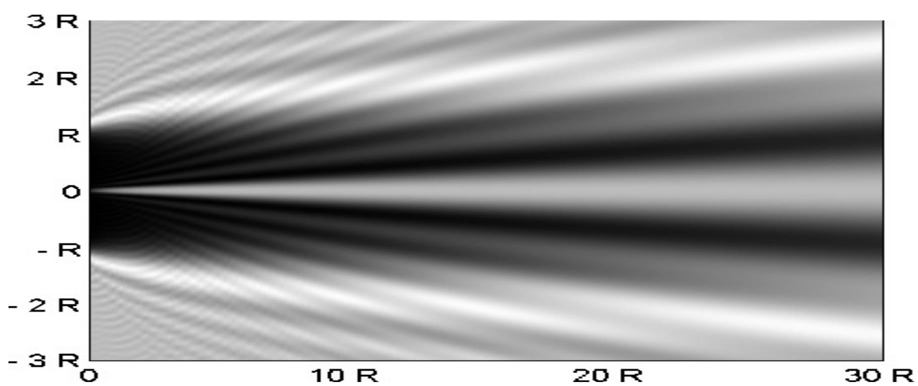

Figure 3: Calculated intensity distributions behind the circular opaque disk in the plane (z, x).

Figure 3 represents the intensity behind an opaque disc of radius $R = 5\mu m$ of a monochromatic wave of light with wavelength $\lambda = \frac{R}{10} = 0.5\ \mu m$.

We see clearly the area corresponding to the geometric shadow, but also the bright spot of Poisson-Arago in the center of the shadow; with the choice of $\lambda = \frac{R}{10}$, this bright spot assumes a great importance.

Newton, who carried out the experiment of the opaque disk using a coin, does not report the presence of fringes within the shadow in his *Opticks*. [2] With $\lambda = 0.5\ \mu m$ and a coin ($R = 1\ cm$), it is difficult to see the bright spot of Arago on a detector placed at 5 m [see Fig. 4]. The radius of the spot is 0.1 mm and just visible to the naked eye. This oversight was to have an unfortunate consequence a century later!

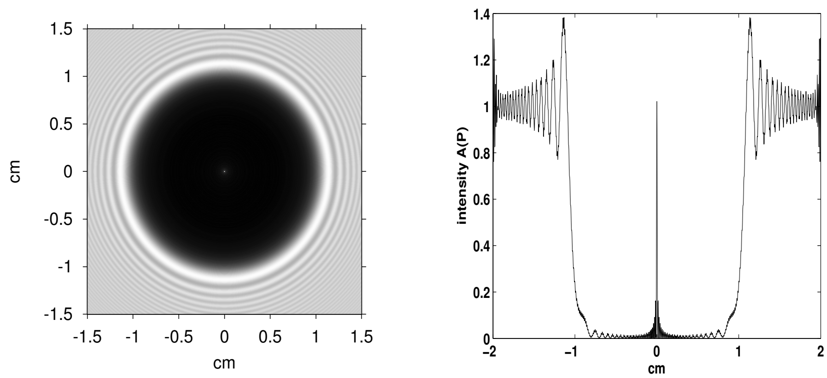

Figure 4: Spot of Poisson-Arago: Intensity distribution behind a coin ($R = 1\ cm$) on a detector placed at 5m.

3 Energy flow lines for a monochromatic wave

Let us consider a monochromatic electromagnetic field $\{\mathbf{E}(\mathbf{r},t),\ \mathbf{B}(\mathbf{r},t)\}$ which is the real part of the complex monochromatic electromagnetic field $\{\mathcal{E}(\mathbf{r},t) = \mathbf{E}_0(\mathbf{r})e^{-i\omega t}, \mathcal{B}(\mathbf{r},t) = \mathbf{B}_0(\mathbf{r})e^{-i\omega t}\}$.

The Poynting vector $\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}$ is the instantaneous rate of energy flow per unit area at a point; $u = \frac{1}{2}(\epsilon_0\mathbf{E}^2 + \frac{1}{\mu_0}\mathbf{B}^2)$ is the instantaneous electromagnetic energy density. Since the optical frequencies are very large (ω is of order $10^{15}\ s^{-1}$), one cannot observe the instantaneous values of any of the rapidly oscillating quantities, but only their time average taken over a time interval which is large compared to the fundamental period $T = \frac{2\pi}{\omega}$. [11, 12]

The energy flow, which is interpreted as the time-averaged Poynting vector, [11, 12] is determined from the real part of the complex Poynting vector $\mathcal{S} = \frac{1}{\mu_0}\mathcal{E}\times\mathcal{B}^*$ and from the energy density of the complex field $\mathcal{U} = \frac{1}{2}(\varepsilon_0\mathcal{E}\mathcal{E}^* + \frac{1}{\mu_0}\mathcal{B}\mathcal{B}^*)$. The time-averaged flux of energy and the time-averaged energy density are given by

$$\langle \mathbf{S} \rangle = \frac{1}{2\mu_0} Re[\mathbf{E}_0 \times \mathbf{B}_0^*], \qquad \langle u \rangle = \frac{1}{4}(\varepsilon_0 \mathbf{E}_0 \mathbf{E}_0^* + \frac{1}{\mu_0} \mathbf{B}_0 \mathbf{B}_0^*). \tag{3}$$

The energy flow lines are obtained by the equation

$$\frac{d\mathbf{r}}{dt} = \frac{\langle \mathbf{S} \rangle}{\langle u \rangle}. \tag{4}$$

For diffraction problems, these energy flow lines have been discussed at length. In 1952, they were calculated numerically for two-dimensional diffraction on a half-plane by Braunbeck and Laukien [10] and recalled in Born and Wolf's textbook [11] p.575-577. In 1976, Prosser [13] proposed an interpretation of diffraction and interference with electromagnetic fields in terms of energy flow lines. These lines are recently demonstrated and discussed in distributions of incoherent light for various two-dimensional situations by Wünscher et al. [14] Their interpretation will be discussed in the following section.

To solve the circular aperture and opaque disc problems, we look for an electromagnetic field $(\mathbf{E}_0, \mathbf{B}_0)$, written in cylindrical coordinates (ρ, φ, z), in the form $\mathbf{E}_0 = \{e_\rho, 0, e_z\}$ and $\mathbf{B}_0 = \{0, b_\varphi, 0\}$, where e_ρ, e_z and b_φ are functions of (ρ, z).

From Maxwell equation $curl\mathcal{B} = \varepsilon_0 \mu_0 \frac{\partial \mathcal{E}}{\partial t}$ we deduce $\mathbf{E}_0 = \frac{ic}{k}[-\frac{\partial}{\partial z} b_\varphi, 0, \frac{\partial}{\partial \rho} b_\varphi]$ and

$$\langle \mathbf{S} \rangle = \frac{1}{2\mu_0} \frac{c\lambda}{2\pi} Im(b_\varphi^* \nabla b_\varphi). \tag{5}$$

From Faraday's law $curl\mathcal{E} = -\frac{\partial \mathcal{B}}{\partial t}$, we show that b_φ verifies the Helmholz equation

$$\Delta b_\varphi(\rho, z) + k^2 b_\varphi(\rho, z) = 0. \tag{6}$$

If we take as boundary conditions $b_\varphi = A_0 e^{ikz}$ on the aperture, then the solution found $(\mathbf{E}_0, \mathbf{B}_0)$ is solution of Maxwell equations.

Taking into account the time-averaged energy conservation law $\nabla \cdot \langle \mathbf{S} \rangle = 0$, we deduce $\varepsilon_0 \mathbf{E}_0 \mathbf{E}_0^* = \frac{1}{\mu_0} \mathbf{B}_0 \mathbf{B}_0^*$ and then $\langle u \rangle = \frac{1}{2\mu_0} b_\varphi b_\varphi^*$.

The energy flow lines are then defined wholly by the wave b_φ

$$\frac{d\mathbf{r}}{dt} = \frac{c\lambda}{2\pi} \frac{Im(b_\varphi^* \nabla b_\varphi)}{b_\varphi b_\varphi^*} \tag{7}$$

and are perpendicular to equal phase surfaces; if $b_\varphi = |b_\varphi| \exp(i\theta)$, $\nabla \theta = Im(b_\varphi^* \nabla b_\varphi)/b_\varphi b_\varphi^*$.

3.1 Energy flow lines for a circular aperture

Figure 5 shows 26 energy flow lines where the initial positions are drawn at random in the circular aperture.

We notice that after a disturbance in the Fresnel zone, lines gradually become straight in the Fraunhofer area, in agreement with the diffracted rays proposed by Newton in Figure 1 for a circular aperture.

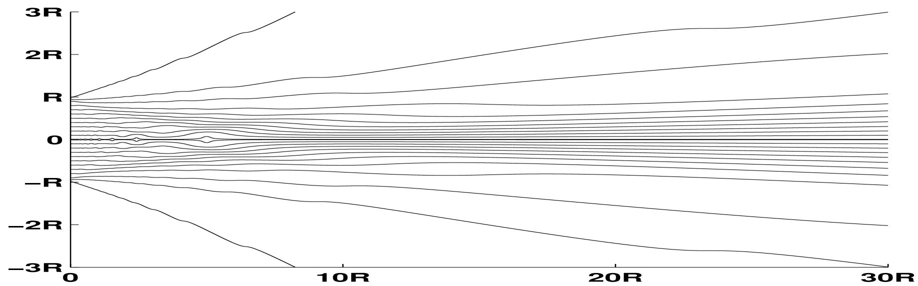

Figure 5: 26 energy flow lines behind the circular aperture.

3.2 Spot of Poisson-Arago and energy flow lines for a circular opaque disk

Figure 6 shows energy flow lines where the initial positions are drawn at random outside the circular opaque disk.

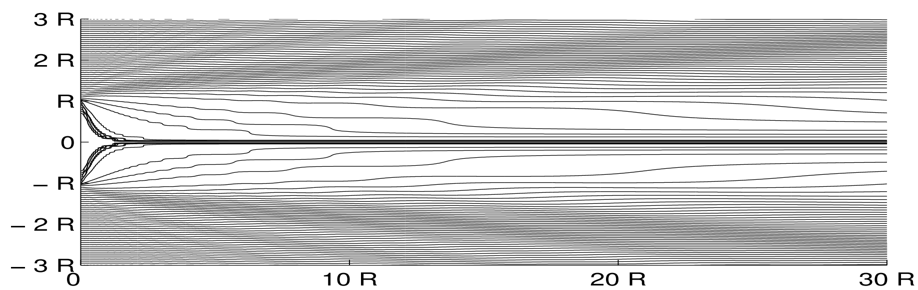

Figure 6: Energy flow lines behind the circular opaque disk.

Using the presence of energy flow lines behind the opaque disc, we propose an explanation of the bright spots of Poisson-Arago in the next section.

3.3 Energy flow lines for Young's double slit experiment

We now complete these numerical simulations by determining the energy flow lines for the Young's double slit experiment. Carried out in 1802 by Thomas Young, some years before Fresnel's theory, this well-know experiment is the first that clearly demonstrates the wave nature of light.[15]

Let us consider a monochromatic plane wave of light ($\lambda = 0.5\ \mu m$) propagating perpendicular to two slits placed in the xy plane, and a detector in a parallel plan at distance z. The slits have a width $d = 5\ \mu m$ along x and infinitely long along y; 2 d is the distance between slits, center to center.

The electromagnetic field $(\mathbf{E}_0, \mathbf{B}_0)$ is function of (x, z) and can be written $\mathbf{B}_0 = [0, 0, b_z]$, $\mathbf{E}_0 = \frac{ic}{k}[\frac{\partial}{\partial y} b_z, -\frac{\partial}{\partial x} b_z, 0]$. The energy flow lines after the slits are then defined by

$$\frac{d\mathbf{r}}{dt} = \frac{c\lambda}{2\pi} \frac{Im(b_z^* \nabla b_z)}{b_z b_z^*}. \tag{8}$$

If the incident wave b_z is of the form $A_0 e^{ikz}$ on the slits, b_z is, after the slits, given by the Fresnel-Kirchhoff solution:

$$b_z(P) = \frac{A_0}{\sqrt{\lambda z}} e^{-i\frac{\pi}{4}} e^{ikz} \int_S e^{\frac{ik(x-x_M)^2}{2z}} dx_M \tag{9}$$

where the integration is taken on the length S of the two slits.

Figure 7 shows 20 energy flow lines where the initial position are drawn at random in the two slits.

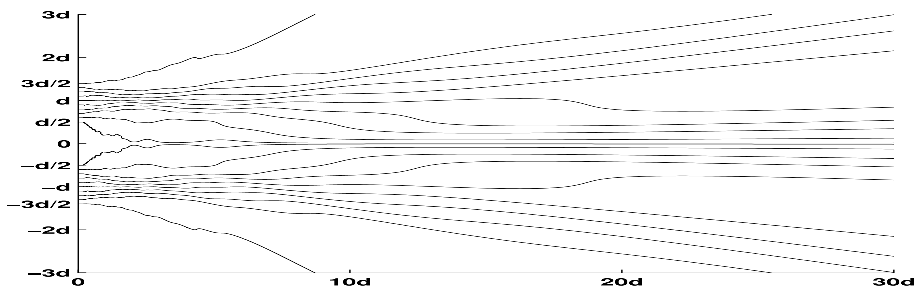

Figure 7: 20 energy flow lines behind the two slits.

4 Interpretation of monochromatic energy flow lines

For monochromatic waves in the vacuum, we can establish a correspondence between these energy flow lines and the diffracted rays proposed by Newton in *Principia* [16] (1687):"*Moreover, the rays of light that are in our air (as lately was discovered by Grimaldi, by the admission of light into a dark room through a small hole, which I have also tried) in their passage near the angles of bodies, whether transparent or opaque (such as the circular and rectangular edges of gold, silver and brass coins, or of knives, or broken pieces of stone or glass), are bent or inflected round those bodies as if they were attracted to them.*"

Can these energy flow lines be interpreted as light rays?

When light is incoherent, we must reject this interpretation as recalled by Wünscher et al. [14] Indeed, when the light is not monochromatic, it must be regarded as a mixture of monochromatic waves as Newton showed in his famous experiments of light decomposition. [2] Each monochromatic wave of white light gives energy flow lines which depend on its wavelength.

The answer is more complex for a monochromatic wave. These energy flow lines are a generalization of the rays of the geometrical optics. Indeed, if we increase the frequency of the light wave towards infinity, the energy flow lines converge towards the straight rays of the geometrical optics. This is shown in Fig. 8 for Young's double slit interference. Since in geometrical optics we speak of the light rays, the energy flow lines for monochromatic waves in vacuum could be called by analogy, the light rays of wave optics. These energy flow lines correspond to the definition of rays of light given by Newton in the begining of his Opticks: " *By the Rays of Light I understand its least Parts, and those as well Successive in the same Lines as Contemporary in several Lines.*"

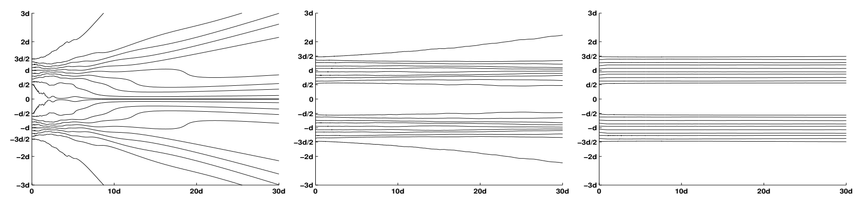

Figure 8: Evolution of the energy flow lines as the frequency increases: $\lambda = 0.5\ \mu m$; $\lambda = 50\ nm$; $\lambda = 5\ nm$.

5 Conclusion

The energy flow lines concept is the simplest answer to the question of the French Academy : "*deduce by mathematical induction, the movements of the rays during their crossing near the bodies*". These lines can correspond to the diffracted rays proposed by Newton, and by analogy to the geometrical optics, they can be also considered as the light rays of wave optics. So the "spot of Poisson-Arago" could be explained by the effect of these rays.

6 Aknowledgements

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